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Periodic Behavior of Cellular Automata

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I propose an explanation of the observation of a globally synchronized behavior of deterministic cellular automata and coupled map lattices, together with local fluctuations.

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In this note I present a qualitative explanation of a striking observation by Chaté and Manneville,[1] confirmed by others,[2] that it is possible for some deterministic cellular automata (CA) to have *on average* a periodic behavior. This phenomenon is remarkable for a number of reasons. First, it is not intuitive at all that CA with a discrete phase space can mimick so closely continuous systems. Furthermore, arguments have been presented[3] tending to show that CA cannot have such a periodic behavior on average, together with small-scale thermal noise. As this relies upon a nucleation mechanism, one may argue that there a periodic behavior (as seen in refs. 1 and 2) is only a transient phenomenon, although the behavior for very long times should be different (as predicted by ref. 3), but never reached in actual computations that are not long enough. This is very unlikely for two reasons: first, the periodic behavior is reached from random initial conditions, and one does not see why random initial conditions would reach a metastable state instead of a stable one; then these CA are defined without any real parameter, everything there is of order 1, so a kind of conspiracy is needed to produce very large transient times. This could happen for one specific rule, but becomes completely unlikely when one realizes that the same has been observed now for four or five CA without any obvious resemblance: for instance, some are on a 3D lattice, others on 4D and 5D

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lattices. As far as I know, the matter is not yet settled, and the present note aims at showing that this phenomenon can be reasonably explained by simple arguments.

For this I propose to see these CA on a mesoscale, replacing then the iteration of discrete variables by an iteration of continuous quantities (= a coupled map lattice) that can be seen as obtained from the CA by local averaging (as done when deriving fluid mechanics for lattice gases, for instance). That is to say that CA share the properties of coupled map lattices, something in full agreement with the property that coupled map lattices may have also a macroscopic periodic behavior together with random fluctuations on a microscale. The core of the argument is as follows: Let $s_i(t+1)=F[s_i(t)]$ be a set of *uncoupled* maps on a lattice where the position index is i, although the discrete time is t. The function $F[\cdot]$ of the real variable s is rather arbitrary at the moment, and could be the familiar quadratic map. As is well known,[4] there are possible choices of this function $F[\cdot]$ such that the iteration yields a random result, such as the map $4s(1-s)$ with s between 0 and 1. If one considers the class of quadratic maps $F[s]=4\lambda s(1-s)$, where λ is a real parameter less than 1, there are values of λ for which the iterated maps is random, but has nevertheless a "window" structure. For instance, in the period-three window, three intervals exist, all included in $]0, 1[$ and denoted as I_1, I_2, I_3, such that if $s(t)$ is inside I_1, then $s(t+1)$ is inside I_2, $s(t+2)$ in I_3, $s(t+3)$ in I_1 again, and so on. Consider now the iteration of this lattice of uncoupled maps for a value of λ such that every map at a vertex of the lattice is in the same interval at the same time, but having a random value within this interval. Consider now the lattice average $S(t)=(1/N)=(1/N)\sum_i s_i(t)$, where the sum $\sum_i$ extends over N vertices, N large. Because of the averaging, this sum will show a periodic behavior in time, with random Gaussian fluctuations of amplitude $N^{-1/2}$ in the large-N limit. This is what was observed by Chaté and Manneville. It is crucial to notice at this point that the Gaussian fluctuations result from the central limit theorem, and this does not imply that the fluctuation of an individual s_i is Gaussian at all: in particular the fluctuation of s_i are strictly bounded. The next step in the reasoning is to assume now that the maps are coupled, that is, one replaces the map $s_i(t+1)=F[s_i(t)]$ by

$$s_i(t+1)=F[s_i(t)]+\eta \sum_{j\in \mathrm{nn}} s_j(t)$$

where the sum is now on the nearest neighbors of i, and where η is a coupling constant. Take η very small. Then the "interaction" term in the coupled iteration will be uniformly small at any time, and so one can guess that the period three will remain stable: this interaction will be too weak to make a particular vertex miss one phase in the succession of intervals I_1, I_2, I_3, I_1, I_2,... in the course of time: these intervals being disconnected, a finite kick is needed to jump from one interval to another one. This is basically how I explain that CA can have a random behavior on a micro-scale and a coherent periodic behavior on average. This explanation leaves open a certain number of questions that I now consider:

1. I supposed that the initial conditions are such that the automaton is synchronized: every vertex is in the same interval I_k, $k = 1, 2$, or 3, at time 0. As I said before, the periodic state is attained from much more general random initial conditions, most likely with vertices distributed evenly among the three intervals. Actually, one can imagine that the interaction is strong enough to put all the vertices in the same state after transients, but weak enough to keep them afterward in the same state. This is possible, because one may argue that the actual amplitude of the interaction term depends numerically on whether a given vertex and its neighbors are all in the same state (then the interaction would have to be small enough to forbid an out-of-phase jump, once the synchronization of the intervals has taken place) or in a different state (then it has to be strong enough to allow the jump to a global synchronized state). This could be made more rigorous by assuming that the width of the intervals I_k is much less than their mutual distance: then the noise resulting from an out-of-phase neighbor would be much larger than the one coming from the fluctuations inside the intervals I_k. The dynamics of the transients is also an interesting question. It is well possible that the domain walls between domains of a different phase move in a definite direction so that the whole automaton gets into a single time phase at the end.

2. It is of interest to understand the source of the difference between fluctuations in equilibrium systems and nonequilibrium ones as considered here. The crucial point is, as I discussed, the fact that, in out-of-equilibrium systems, fluctuations may have a strictly bounded amplitude, although they are unbounded in equilibrium systems, even though the large excursions are very unlikely in general. To have a model in mind, one could think of a system of Duffing oscillators coupled on a lattice. Let $X_i(t)$ be the coordinate at the site i and time t. The equations of motion for this system are given by

$$\frac{d^2X_i(t)}{dt^2} = X_i(t)[1 - X_i^2(t)] + \eta \sum_{j \in nn} X_j(t)$$

In the absence of coupling ($\eta = 0$) this is integrable in terms of elliptic functions, and the energy of each oscillator is conserved. Suppose that this energy is low enough to keep the system in one of the energy wells either near $X = +1$ or -1. When uncoupled to its neighbors, a Duffing oscillator will never jump to the other well once it started inside one well. On the contrary, if it is coupled, however weakly, to the neighbors (η not zero but small), it will have a chance in the course of time to gather the energy of a large number of neighbors and then jump to the other side of the potential well, if one assumes this system to be ergodic. This emphasizes that mechanical systems may have much larger fluctuations than CA, because of the conservation of energy.

REFERENCES

1. H. Chaté and P. Manneville, *Europhys. Lett.* **14**:409 (1991).
2. J. A. C. Gallas, P. Grassberger, H. J. Herrmann, and P. Ueberholz, Noisy collective behavior in deterministic cellular automata, IFK Jülich preprint (1991).
3. T. Bohr, G. Grinstein, Yu He, and C. Jayarprakash, *Phys. Rev. Lett.* **58**:2155 (1987).
4. Hao Bai-lin, *Chaos II* (World Scientific, Singapore, 1990).

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